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F-term equations near Gepner points

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ABSTRACT: We study marginal deformations of B-type D-branes in Landau-Ginzburg orbifolds. The general setup of matrix factorizations allows for exact computations of F-term equations in the low-energy effective theory which are much simpler than in a corresponding geometric description. We present a number of obstructed and unobstructed examples in detail, including one in which a closed string modulus is obstructed by the presence of D-branes. In a certain example, we find a non-trivial global structure of the BRST operator on the moduli space of branes.

KEYWORDS: D-branes, Superstring Vacua, Conformal Field Models in String Theory.

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Contents

1. Introduction

D-branes in non-trivial Calabi-Yau(CY) backgrounds are interesting to study and find many applications throughout string theory. The subject has been investigated intensively over the past half decade, with many remarkable results. One part of the story that is still less understood are the properties of D-brane moduli spaces, both at the local and in particular at the global level. Some progress has been obtained in the context of non-compact toric models, but the computation of D-brane superpotentials for a generic compact CY model is in general still missing, despite considerable interest in particular from the phenomenological point of view.

A large class of string compactification backgrounds admit in some part of closed string moduli space a description as Landau-Ginzburg orbifold (LG) models. In the bulk, such LG models are specified by the choice of a quasihomogeneous polynomial W as worldsheet superpotential. Since [1], finding useful B-type boundary conditions in Landau-Ginzburg models was a nagging problem (see for example [2, 3, 4, 5]). Recently, M. Kontsevich has given a description based on a so-called matrix factorization of W [6] which is found to be useful in more recent works [7]–[16]. It can be succinctly written as the equation

$$Q^2 = W \cdot \mathrm{id}\,, \tag{1.1}$$

on a matrix Q with polynomial entries which encodes boundary interactions on the worldsheet.

The purpose of this note is to open up this window on the possibility of studying D-brane moduli spaces and D-brane superpotentials in this very simple algebraic setup. We will firstly discuss some general aspects of the deformation problem of D-branes in their description as matrix factorizations. We focus on marginal deformations which are relevant for the moduli problem. (Relevant deformations, important for discussion on (in)stability and decays, were studied in [14, 15].) Secondly, we will apply the technology to a number of relevant examples. We will start with the case $\hat{c} = 1$, corresponding geometrically to an elliptic curve. We will show how the moduli space of its matrix factorizations is naturally the torus itself. Results from the mathematical literature [17] suggest that the problem of finding all matrix factorizations for the torus case is essentially solved. In this example, we also find that the matrix Q, which is a part of the BRST operator in the context of string field theory, transforms non-trivially as we move around in the moduli space. We then turn to the physically interesting case $\hat{c} = 3$. We study in detail the quintic model and the behavior of its rational branes under open and closed string deformations. One spinoff of our results is the reconciliation, via mirror symmetry, of the behavior of A-branes wrapped on $\mathbb{RP}^3$ inside the quintic over Kähler moduli space. We will also find an example where a certain deformation of closed string moduli is obstructed by the presence of a D-brane. Such a phenomenon is known to be possible on general grounds but a concrete example had not been found in the context of 4d $\mathcal{N} = 1$ supersymmetry.

Note: this publication was prompted by the preprint [18], which appeared while we were contemplating publication of our results, and in which the idea of using B-type LG branes for computing spacetime superpotentials is also discussed.

2. Generalities

We consider two-dimensional $\mathcal{N} = 2$ Landau-Ginzburg theories of relevance for superstring compactifications (LG models). We propose to study conformally invariant B-type boundary conditions as worldsheet descriptions of D-branes in such models.

2.1 The bulk

The construction of such LG models begins with picking a superpotential W, which is a holomorphic function of r chiral field variables x_i. Since we require a conformally invariant

IR fixed point, we take W to be a quasihomogeneous polynomial [19, 20, 21, 22], of degree denoted by H, where each x_i has weight w_i, i.e.,

$$W(\lambda^{w_1}x_1,\dots,\lambda^{w_r}x_r)=\lambda^H W(x_1,\dots,x_r) \qquad \text{for } \lambda\in\mathbb{C}\,. \tag{2.1}$$

The central charge of the theory at the IR fixed point is given by

$$\hat{c}=\sum_i\left(1-\frac{2w_i}{H}\right)=\sum_i\left(1-\frac{2}{h_i}\right). \tag{2.2}$$

We will mostly assume that all w_i divide H, so that h_i are integer, and there exists a Fermat point in the moduli space of conformal theories, at which W takes the form,

$$W=x_1^{h_1}+\cdots+x_r^{h_r}\,. \tag{2.3}$$

At this Fermat point, the IR fixed point theory is rational and equivalent to a tensor product of $\mathcal{N}=2$ minimal models, generically refered to as a Gepner model [23].

The chiral (c,c) ring [24] of such an LG theory is given by

$$\mathcal{R}=\mathbb{C}[x_i]/\langle dW\rangle \tag{2.4}$$

and because of (2.1) is graded by the vector $\mathrm{U}(1)$ R-charge q, which is normalized such that W has charge 2,

$$q_i=q(x_i)=\frac{2}{h_i}\,. \tag{2.5}$$

Let us also assume that the central charge (2.2), which measures the number of compactified dimensions, is integer. As is well-known (see, e.g., [25]) we can then project onto integral $\mathrm{U}(1)$ charge by orbifolding by the global symmetry group $\Gamma_0\cong\mathbb{Z}_H$ whose generator acts by

$$x_i\mapsto\omega^{w_i}x_i \tag{2.6}$$

with $\omega^H=1$. In the RCFT at the Fermat point, this orbifold operation is most conveniently phrased in terms of simple-current extensions, but we will not use this language here.

Orbifolding thus projects the (c,c) ring (2.4) onto integrally charged fields. Twisted sectors contain the (a,c) ring as well as possibly additional elements of the (c,c) ring. We often enlarge our orbifold group Γ to contain other global symmetries as well. The maximal example is the one $\widehat{\Gamma}_0$ for Greene-Plesser mirror [26] of Γ_0, $x_i\to\omega_i x_i$ where $\omega_1^{h_1}=\cdots=\omega_r^{h_r}=\omega_1\cdots\omega_r=1$. More generally, Γ is included in $\widehat{\Gamma}_0$ and includes Γ_0.

2.2 B-type boundary conditions

As explained in [7, 9, 10, 12] the problem of finding boundary conditions preserving B-type $\mathcal{N}=2$ supersymmetry in an LG model becomes equivalent to finding a matrix $Q=Q(x_i)$ satisfying (1.1) [6]. To explain (1.1) more precisely, we are looking for a square matrix Q whose entries are polynomials in x_i, such that there exists a grading operator σ (a matrix with scalar entries, $\sigma^2=1$) satisfying

$$\sigma Q+Q\sigma=0\,. \tag{2.7}$$

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Q can be thought of as acting on a $\mathbb{Z}_2$ graded $\mathbb{C}[x_i]$-module (with grading provided by σ), which we will denote by N. The r.h.s. of equation (1.1) is interpreted as the bulk LG potential W times the identity matrix. Equivalently, by diagonalizing σ, we are looking for a pair of matrices f and g such that

$$Q = \begin{pmatrix} 0 & f \\ g & 0 \end{pmatrix} \qquad \left[\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right], \tag{2.8}$$

in terms of which (1.1) becomes

$$fg = gf = W \cdot \mathrm{id}\,. \tag{2.9}$$

The physical interpretation of this formalism is that Q represents a tachyon configuration between a stack of spacefilling branes and a stack of spacefilling antibranes, corresponding to the positive and negative eigenspaces of σ, respectively.

Because of the form (2.9), a solution of (1.1) is also known as a matrix factorization in the mathematical literature. We will use this terminology as well as referring to the triple (N, σ, Q) as a B-type LG brane, often dropping Q and σ.

Generally, we do not definitely want to fix the dimension of Q (which is even and twice the dimensions of f and g). This is because we actually want to identify solutions which differ by the addition of a "trivial brane-antibrane pair" corresponding to

$$Q_{\text{trivial}} = \begin{pmatrix} 0 & 1 \\ W & 0 \end{pmatrix}. \tag{2.10}$$

In other words we want to divide the space of solutions of (1.1) by the equivalence relation

$$Q \equiv Q \oplus Q_{\text{trivial}}\,. \tag{2.11}$$

We also want to identify, of course, solutions of (1.1) which differ only by a similarity transformation

$$Q \mapsto UQU^{-1} \tag{2.12}$$

where U is an invertible matrix with polynomial entries.

As in the bulk, one expects that the boundary interactions defined by such a matrix factorization will flow to a conformally invariant boundary field theory in the IR if there exists a conserved U(1) R-charge. Since the U(1) charge of the x_i is fixed from the bulk, the only freedom we have is the action on the CP spaces. Thus, we require the existence of a matrix S such that Q has definite charge under it,

$$e^{i\alpha S} Q\big(e^{i\alpha q_i} x_i\big) e^{-i\alpha S} = e^{i\alpha q(Q)} Q(x_i)\,. \tag{2.13}$$

It is clear from (1.1) that if we normalize W to have charge 2, $q(Q) = 1$. The condition (2.13) is the boundary equivalent of (2.1), and S is part of the data specifying a conformally invariant B-type boundary condition.

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The spaces of boundary chiral fields from one brane (M, P, ρ) to another such brane (N, Q, σ) (for the same bulk superpotential W) is obtained from the space

$$\mathrm{Hom}_{\mathbb{C}[x_i]}(M, N)\,, \tag{2.14}$$

by taking the cohomology of the operator D defined by

$$D(\Phi) = Q\Phi + \sigma\Phi\rho P\,, \tag{2.15}$$

for an arbitrary $\Phi = \Phi(x_i)$ (a matrix with polynomial entries). We will denote the space by

$$H^*(M, N) = \mathrm{Ker}(D)/\,\mathrm{Im}(D)\,. \tag{2.16}$$

In general, this space is $\mathbb{Z}_2$ graded by σ and ρ, i.e., homogeneous elements satisfy

$$\sigma\Phi\rho = (-1)^{\Phi}\Phi\,. \tag{2.17}$$

However, in the conformally invariant case, we have another, finer, (in general non-integral!) grading by $\mathrm{U}(1)$ R-charge. Denoting the R-charges associated with (M, P, ρ) and (N, Q, σ) by R and S respectively, we can contemplate homogeneous elements satisfying

$$e^{i\alpha S}\,\Phi\big(e^{i\alpha q_i}x_i\big)\,e^{-i\alpha R} = e^{i\alpha q(\Phi)}\Phi(x_i)\,. \tag{2.18}$$

We also note that imposing unitarity of the worldsheet theory requires $0 \le q \le \hat{c}$ for all chiral fields [24]. Clearly, this is a condition on the matrices R and S in addition to (2.13). Similarly to the bulk, charge conjugation invariance of the open string RR sector translates into "Serre duality" for boundary chiral fields. We will denote it by $\dagger$. It maps $\Phi \in H^*(M, N)$ to $\Phi^\dagger \in H^*(N, M)$ and satisfies

$$q(\Phi^\dagger) = \hat{c} - q(\Phi)\,. \tag{2.19}$$

Looking back at the bulk, we see that the next step in the construction is orbifolding. On the boundary, this is again implemented by the choice of an action on the CP spaces. Choosing a representation of Γ for each brane (M, σ, Q), we impose equivariance on the factorization Q

$$\gamma\, Q\big(\gamma(x_i)\big)\gamma^{-1} = Q(x_i) \qquad \text{for every } \gamma \in \Gamma\,. \tag{2.20}$$

Open string fields are projected similarly. It must be that after such a projection all fields from a brane to itself have integer R-charge, which upon mod2 reduction is the same as the $\mathbb{Z}_2$ grading.

2.3 Deformations and obstructions

Let us now fix one such B-brane (M, σ, Q) in some appropriately orbifolded bulk theory with bulk superpotential W. We want to ask the following questions

(i) Can we deform Q, holding W fixed?

(ii) If we deform W, is there a deformation of Q that satisfies (1.1)?

If we restrict to infinitesimal deformations, the answer to question (i) is simple. Infinitesimal deformations correspond to elements

$$\Phi \in H^1(M, M)\,, \tag{2.21}$$

since clearly the deformed $Q(\varphi) = Q + \varphi\Phi$ satisfies

$$\big(Q(\varphi)\big)^2 = Q^2 + \varphi\{Q, \Phi\} + \varphi^2\Phi^2 = W + \mathcal{O}(\varphi^2)\,, \tag{2.22}$$

i.e., it squares to W to first order in φ. It is easy to see that we cannot remove $\varphi\Phi$ from $Q(\varphi)$ by a gauge transformation, unless Φ is trivial in $H^1(M, M)$.

Let us try to continue the first order deformation (2.22) to higher order in φ, i.e., we write

$$Q(\varphi) = \sum_n \varphi^n Q_n\,, \tag{2.23}$$

where $Q_0 = Q$ we started with, $Q_1 = \Phi$, and all Q_n have odd degree and R-charge 1. Imposing $\big(Q(\varphi)\big)^2 = W$ then leads at order n to the equation

$$\{Q_0, Q_n\} = -\sum_{k=1}^{n-1} Q_k Q_{n-k}\,. \tag{2.24}$$

Assume that we have found a deformation up to order $n-1$. The r.h.s. of (2.24) is then Q_0 closed,

$$\left\{Q_0, \sum_{k=1}^{n-1} Q_k Q_{n-k}\right\} = -\sum_{k=1}^{n-1}\left(\sum_{l=1}^{k-1} Q_l Q_{k-l} Q_{n-k} - \sum_{l=1}^{n-k-1} Q_k Q_l Q_{n-k-l}\right) = 0\,. \tag{2.25}$$

Thus, we can solve to order n unless the r.h.s. of (2.24) is in the cohomology of $Q_0 = Q$. Our problem being $\mathbb{Z}$-graded, the r.h.s. has degree 2 so the possible obstructions lie in $H^2(M, M)$.

A simple consequence of these considerations is the dependence of the deformation problem on the dimension, or central charge $\hat{c}$ of our model. For $\hat{c} = 1$ (compactification on a torus), H^p vanishes for $p > 1$, so the deformation problem is never obstructed. We will see explicitly in a later section how this is implemented in practice.

The case $\hat{c} = 3$ is the most interesting one from the physics point of view. We note that in that case, Serre duality (2.19) implies that to every first order deformation $\Phi \in H^1(Q)$, there exists a corresponding obstruction $\Phi^\dagger \in H^2(Q)$. "Generically", one would expect that all obstructions appear in the deformation problem. As we will see, however, this does not mean that there are no finite boundary deformations for $\hat{c} = 3$. Instead, it can happen that the boundary obstructions actually serve to lift a previously marginal bulk deformation!

Thus turning to problem (ii), we consider a bulk deformation $W \to W + \psi\Psi$, where Ψ is a polyonmial in x_i of total degree H, i.e., left-right U(1)-charge $q_L = q_R = 1$. Multiplying Ψ with the identity matrix transforms it into a boundary field with R-charge $q = q_L + q_R = 2$. Obviously, $\Psi \cdot \mathrm{id}$ is Q-closed, so with the ansatz

$$Q(\psi) = \sum \psi^n Q_n \tag{2.26}$$

and $Q_0 = Q$, we can solve (1.1) to first order in ψ only if

$$\{Q_0, Q_1\} = \Psi\,\mathrm{id} \tag{2.27}$$

i.e., $[\Psi] = 0 \in H^2(Q)$. In this case, we obtain at order n in ψ the condition

$$\{Q_0, Q_n\} = -\sum_{k=1}^{n-1} Q_k Q_{n-k}\,. \tag{2.28}$$

Superficially, this looks like (2.24). And indeed, even though Q_1 is not closed, the argument we gave above still goes through (because Ψ commutes with everybody on the boundary) to show that if we have solved to order $n-1$, we can solve to order n if and only if the r.h.s. of (2.28) is trivial in $H^2(Q)$.

On the other hand, if Ψ is non-trivial in $H^2(Q)$, we will not be able to solve (1.1), at least not with the ansatz (2.26). We will see in an example that what can happen in that case is that there is a first order boundary deformation Φ which squares to Ψ. In that case we obtain two families of solutions

$$\big(Q + \varphi\Phi\big)^2 = W + \psi\Psi \tag{2.29}$$

for $\varphi^2 = \psi$. More generally, Ψ might appear as an obstruction to a boundary deformation $\varphi\Phi$ at some higher order $n > 2$. In this case, we obtain a polynomial constraint between ψ and φ of order n.

Still, it is possible that in the general case in which a bulk field Ψ restricts on the boundary to an element of $H^2(Q)$, there is no corresponding boundary deformation that yields a solution of (1.1). In that case, that bulk deformation is obstructed by the presence of the D-brane. We will find such an example in section 6.1.

On the other hand, it should also be noted that not all obstructions on the boundary can be lifted by deforming the bulk. This is because, quite simply, not all boundary fields of charge 2 are proportional to the identity matrix and can be moved to the bulk.

2.4 F-terms and superpotentials

$\mathcal{N} = 2$ superconformal symmetry and charge intgrality on the worldsheet is the condition for spacetime $\mathcal{N} = 1$ supersymmetry at string tree level, described by the D-term and F-term equations in the low energy effective supergravity. We have discussed in the previous subsection that studying the deformation problem for B-branes naturally leads to some holomorphic constraints on the open and closed deformation parameters φ and ψ. These deformation parameters become $\mathcal{N} = 1$ chiral fields in the low-energy theory.

The constraints on the fields φ and ψ are then naturally interpreted as F-term equations. On general grounds, see e.g., [27], F-terms are related to $\mathcal{N} = 2$ supersymmetry on the worldsheet, and since (1.1) is equivalent to preserving $\mathcal{N} = 2$ worldsheet supersymmetry, we conclude that these constraints on φ and ψ we find from studying brane deformations are all F-term constraints in the tree level low-energy theory. On the other hand, the requirement of conformal invariance, or, equivalently, conditions (2.13) and (2.18) on the R-charge, are related to D-terms in spacetime [27, 28].

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Given the F-term constraints, it is natural to ask whether one can integrate them to obtain an $\mathcal{N}=1$ spacetime superpotential $\mathcal{W}(\varphi,\psi)$ governing the dynamics of light open and closed string fields. One requirement is certainly that finding the locus of $\mathcal{W}=d\mathcal{W}=0$ must correspond to solving (1.1). However, it is also important that $\mathcal{W}$ be expressed in the natural "flat variables", e.g., in order for it to be useful for mirror symmetry. Moreover, there are certain global requirements on $\mathcal{W}$ that must be taken into account, such as that it contain all fields that can become massless at some point in the moduli space, as well as that it take value in the right bundle over configuration space. We will mention some of these global conditions in the examples below.

General prescriptions for the computation of $\mathcal{W}$ from the topological string theory have been discussed, e.g., in [29, 30, 31, 15] (the underlying mathematical literature is [32]). The results of [33, 10] on topological correlators in LG model also should be useful.

3. Constructions

3.1 Minimal models

B-branes in the $\mathcal{N}=2$ minimal models have been studied from the point of view that we take here in particular in [9, 11, 14, 15].

In the minimal model with $W = x^h$, we denote the B-brane associated with the factorization $W = x^n x^{h-n}$ by (M_n, Q_n), where M_n is a rank two free module and

$$Q_n = Q_n(x) = \begin{pmatrix} 0 & x^n \\ x^{h-n} & 0 \end{pmatrix}. \tag{3.1}$$

The fermionic and bosonic operators between M_{n_1} and M_{n_2} are given by

$$\phi^1_{n_1,n_2,j}(x) = \begin{pmatrix} 0 & x^{\frac{n_1+n_2}{2}-j-1} \\ -x^{h-\frac{n_1+n_2}{2}-j-1} & 0 \end{pmatrix}, \tag{3.2}$$

and

$$\phi^0_{n_1,n_2,j}(x) = \begin{pmatrix} x^{j-\frac{n_1-n_2}{2}} & 0 \\ 0 & x^{j+\frac{n_1-n_2}{2}} \end{pmatrix}, \tag{3.3}$$

where

$$j = \frac{|n_1-n_2|}{2}, \frac{|n_1-n_2|}{2}+1, \ldots, \min\left\{\frac{n_1+n_2}{2}-1, h-\frac{n_1+n_2}{2}-1\right\}.$$

The R-charge matrix is given by

$$R_n = \begin{pmatrix} \frac{1}{2}-\frac{n}{h} & 0 \\ 0 & -\frac{1}{2}+\frac{n}{h} \end{pmatrix}.$$

This is determined by the invariance of the boundary interaction [14], up to a shift by matrix proportional to the identity. This choice is such that the Serre duality holds: The fermionic field $\phi^1_{n_1,n_2,j}(x)$ and the bosonic field $\phi^0_{n_1,n_2,j}(x)$ have R-charges $1-\frac{2+2j}{h}$ and $\frac{2j}{h}$ respectively, and $\phi^1_{n_1,n_2,j}(x)$ and $\phi^0_{n_2,n_1,j}(x)$ are indeed Serre dual of each other.

The $\mathbb{Z}_h$ symmetry $x \to \omega x$, $\omega^h = 1$, induces actions on the Chan-Paton factor M_n. They are labeled by a mod $2h$ integer m such that $n+m$ is even;

$$\gamma_{n,m}(\omega) = \begin{pmatrix} \omega^{-\frac{n+m}{2}} & 0 \\ 0 & \omega^{\frac{n-m}{2}} \end{pmatrix}. \tag{3.4}$$

We denote by $M_{n,m}$ the B-brane M_n equipped with this $\mathbb{Z}_h$-action. The $\mathbb{Z}_h$ symmetry acts on the fields between M_{n_1,m_1} and M_{n_2,m_2} by $\phi(x) \mapsto \gamma_{n_2,m_2}(\omega)\phi(\omega x)\gamma_{n_1,m_1}(\omega)^{-1}$. In particular, the chiral fields are transformed as follows

$$\phi^0_{n_1,n_2,j}(x) \mapsto \omega^{j+\frac{m_1-m_2}{2}}\phi^0_{n_1,n_2,j}(x)\,, \qquad \phi^1_{n_1,n_2,j}(x) \mapsto \omega^{-j-1+\frac{m_1-m_2}{2}}\phi^1_{n_1,n_2,j}(x). \tag{3.5}$$

3.2 Tensor products

As the first step toward Gepner model, we construct the tensor products of the above elementary factorizations. By "tensor product", we here mean in the graded sense. Slightly formally, this graded tensor product differs from the ordinary tensor product only in that composition of maps respects the grading. If $\otimes$ denotes the ordinary tensor product and $\odot$ the graded version, then for graded vector spaces (M_1,ρ_1) and (M_2,ρ_2), we have simply $(M_1,\rho_1)\odot(M_2,\rho_2) \cong (M_1\otimes M_2, \rho_1\otimes\rho_2)$. However for morphisms $\phi_i : (M_i,\rho_i) \longrightarrow (N_i,\sigma_i)$, we have

$$\phi_1 \odot \phi_2 = \phi_1\rho_1^{\phi_2} \otimes \phi_2\,, \tag{3.6}$$

such that composition satisfies

$$(\phi_1 \odot \phi_2)(\psi_1 \odot \psi_2) = (-1)^{\phi_2\psi_1}\phi_1\psi_1 \odot \phi_2\psi_2 \tag{3.7}$$

(for homogeneous maps ϕ_2 and ψ_1).

Explicitly, given a matrix factorization (N_1,σ_1,Q_1) of W_1 and (N_2,σ_2,Q_2) of W_2, we have the graded tensor product

$$\begin{aligned}(N_1,\sigma_1,Q_1)\odot(N_2,\sigma_2,Q_2) &= (N_1\odot N_2, \sigma_1\odot\sigma_2, Q_1\odot 1 + 1\odot Q_2) &(3.8)\\ &= (N_1\otimes N_2, \sigma_1\otimes\sigma_2, Q_1\otimes 1 + \sigma_1\otimes Q_2)\,. &(3.9)\end{aligned}$$

It is trivial to check that $Q = Q_1\otimes 1 + \sigma_1\otimes Q_2$ squares to $W = W_1 + W_2$.

One can check in general [13] that for two such tensor products $(M,\rho,P) = (M_1,\rho_1,P_1)\odot(M_2,\rho_2,P_2)$ and $(N,\sigma,Q) = (N_1,\sigma_1,Q_1)\odot(N_2,\sigma_2,Q_2)$ of $W = W_1 + W_2$, we have the Künneth formula on the cohomologies,

$$H^*(M_1\odot M_2, N_1\odot N_2) \cong H^*(M_1,N_1)\odot H^*(M_2,N_2)\,. \tag{3.10}$$

It is also clear that the general considerations concerning R-charges and orbifold projection described in subsection 2.2 are compatible with taking graded tensor products. In particular, (3.10) is graded by R-charge in that context. In the remainder of the paper, we will revert to denoting the tensor product of matrix factorizations by $\otimes$.

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3.3 Orbifolds

Branes in the orbifold model are labeled by $\mathbf{n} = (n_1, \ldots, n_r)$ for the tensor product brane and $\mathbf{m} = (m_1, \ldots, m_r)$ which specifies the orbifold group action on the Chan-Paton factor. The chiral fields between two such branes are simply the fields which are invariant under the orbifold group action [14, 13].

Let us consider the boundary preserving sector, $\mathbf{n}_1 = \mathbf{n}_2 = \mathbf{n}$, $\mathbf{m}_1 = \mathbf{m}_2$. The chiral field $\otimes_{i=1}^{n} \phi^{s_i}_{n_i,n_i,j_i}(x_i)$ transforms under $(\omega_1, \ldots, \omega_r) \in \Gamma$ by the phase

$$\prod_{s_i=0} \omega_i^{j_i} \prod_{s_i=1} \omega_i^{-j_i-1} \,. \tag{3.11}$$

See the action (3.5). Since Γ always includes the element $(\omega_1, \ldots, \omega_r)$ with $\omega_i = e^{\frac{2\pi i}{h_i}}$, $\sum_{s_i=0} \frac{j_i}{h_i} - \sum_{s_i=1} \frac{j_i+1}{h_i}$ must be an integer for an invariant field $\otimes_i \phi^{s_i}_{n_i,n_i,j_i}(x_i)$. The R-charge of such a field is

$$q = \sum_{s_i=0} \frac{2j_i}{h_i} + \sum_{s_i=1} \left(1 - \frac{2j_i+2}{h_i}\right) = \#\{i|s_i = 1\} + 2\left(\sum_{s_i=0} \frac{j_i}{h_i} - \sum_{s_i=1} \frac{j_i+1}{h_i}\right), \tag{3.12}$$

which is indeed an even integer (resp. odd integer) if the field is bosonic (resp. fermionic). We also note that the Serre dual is obtained by flipping the s_i-label,

$$\otimes_i \phi^{s_i}_{n_i,n_i,j_i}(x_i) \overset{\text{Serre}}{\longleftrightarrow} \otimes_i \phi^{1-s_i}_{n_i,n_i,j_i}(x_i) \,. \tag{3.13}$$

Γ-invariance of the two sides are equivalent since $(\omega_1, \ldots, \omega_r) \in \Gamma$ obeys $\omega_1 \cdots \omega_r = 1$.

In what follows, we usually drop the $\mathbf{m}$-labels since we mainly consider the sectors with $\mathbf{m}_1 = \mathbf{m}_2$.

4. The torus

In this section, we consider matrix factorizations of the LG potential for the two-dimensional torus,

$$W = x_1^3 + x_2^3 + x_3^3 + \psi x_1 x_2 x_3 \tag{4.1}$$

with modulus ψ. All our constructions will be $\mathbb{Z}_3$ equivariant, but we do not make this explicit. As mentioned in section 2.3, first order deformations should not be obstructed since there is no obstruction class in the model with $\hat{c} = 1$. Even in such a case, whether the series $\sum_n \varphi^n Q_n$ has a finite radius of convergence is a non-trivial problem [34]. We will in fact find finite deformations of rational branes, both at $\psi = 0$ and also for finite ψ, and observe that a non-trivial global geometry of the moduli space of branes emerges.

Our results depend crucially on a mathematical literature [17] in which matrix factorizations of (4.1) (for $\psi = 0$) have been studied and, as we understand, classified. This will also provide us to find a clue on geometric interpretation of the Landau-Ginzburg branes.

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4.1 A family of matrix factorizations

Here, we consider a particular family of matrix factorizations which reduces in a limit to the tensor product of minimal model branes. This solution is obtained by utilizing results from [17]. Consider the matrix

$$A = \begin{pmatrix} \alpha x_1 & \beta x_3 & \gamma x_2 \\ \gamma x_3 & \alpha x_2 & \beta x_1 \\ \beta x_2 & \gamma x_1 & \alpha x_3 \end{pmatrix}. \tag{4.2}$$

We see that

$$\det A = (\alpha^3 + \beta^3 + \gamma^3) x_1 x_2 x_3 - \alpha\beta\gamma (x_1^3 + x_2^3 + x_3^3) \tag{4.3}$$

which is equal to λW with

$$\lambda = -\alpha\beta\gamma \tag{4.4}$$

if and only if

$$\alpha^3 + \beta^3 + \gamma^3 + \psi\alpha\beta\gamma = 0\,. \tag{4.5}$$

Let B be the adjoint of A up to a factor,

$$B := \frac{1}{\lambda}\,\mathrm{adj}(A) \tag{4.6}$$

$$= -\frac{1}{\alpha\beta\gamma} \begin{pmatrix} \alpha^2 x_2 x_3 - \beta\gamma x_1^2 & \gamma^2 x_1 x_2 - \alpha\beta x_3^2 & \beta^2 x_1 x_3 - \alpha\gamma x_2^2 \\ \beta^2 x_1 x_2 - \alpha\gamma x_3^2 & \alpha^2 x_2 x_3 - \beta\gamma x_2^2 & \gamma^2 x_2 x_3 - \alpha\beta x_1^2 \\ \gamma^2 x_1 x_3 - \alpha\beta x_2^2 & \beta^2 x_2 x_3 - \alpha\gamma x_1^2 & \alpha^2 x_1 x_2 - \beta\gamma x_3^2 \end{pmatrix}. \tag{4.7}$$

Then we find

$$AB = BA = W\,\mathrm{id}\,, \tag{4.8}$$

as long as (α, β, γ) obeys (4.5) and $\alpha\beta\gamma$ is non-zero. This matrix factorization becomes singular as $\lambda \to 0$, but we can take the limit by using a trick. Let us consider $\alpha \to 0$, $\beta/\gamma \to -1$ as an example. We begin by adding a trivial brane-antibrane pair

$$f = \begin{pmatrix} -\frac{1}{\alpha} W & 0 \\ 0 & A \end{pmatrix} \qquad g = \begin{pmatrix} -\alpha & 0 \\ 0 & B \end{pmatrix} \tag{4.9}$$

and make some elementary transformations such as to remove the singular part of B. The point is that the singular piece of B is of the form

$$B \sim \frac{1}{\alpha} \begin{pmatrix} x_1^2 & x_1 x_2 & x_1 x_3 \\ x_1 x_2 & x_2^2 & x_2 x_3 \\ x_1 x_3 & x_2 x_3 & x_3^2 \end{pmatrix} + \text{regular} = \frac{1}{\alpha} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \cdot \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} + \text{regular} \tag{4.10}$$

So we consider

$$f \to \tilde{f} = U^{-1T} f U^{-1} \qquad g \to \tilde{g} = U g U^T \tag{4.11}$$

where

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -x_1/\alpha & 1 & 0 & 0 \\ -x_2/\alpha & 0 & 1 & 0 \\ -x_3/\alpha & 0 & 0 & 1 \end{pmatrix} \tag{4.12}$$

and find

$$\tilde{g} = \begin{pmatrix} -\alpha & x_1 & x_2 & x_3 \\ x_1 & -\frac{\alpha}{\beta\gamma}x_2x_3 & \frac{x_3^2}{\gamma} - \frac{1}{\alpha}\left(1+\frac{\gamma}{\beta}\right)x_1x_2 & \frac{x_2^2}{\beta} - \frac{1}{\alpha}\left(1+\frac{\beta}{\gamma}\right)x_1x_3 \\ x_2 & \frac{x_3^2}{\beta} - \frac{1}{\alpha}\left(1+\frac{\beta}{\gamma}\right)x_1x_2 & -\frac{\alpha}{\beta\gamma}x_1x_3 & \frac{x_1^2}{\gamma} - \frac{1}{\alpha}\left(1+\frac{\gamma}{\beta}\right)x_2x_3 \\ x_3 & \frac{x_2^2}{\gamma} - \frac{1}{\alpha}\left(1+\frac{\gamma}{\beta}\right)x_1x_3 & \frac{x_1^2}{\beta} - \frac{1}{\alpha}\left(1+\frac{\beta}{\gamma}\right)x_2x_3 & -\frac{\alpha}{\beta\gamma}x_1x_2 \end{pmatrix} \tag{4.13}$$

This is nonsingular in the limit $\alpha \to 0$, $\beta/\gamma \to -1$. In fact, it is easy to see from (4.5)

$$\beta + \gamma = \frac{\psi}{3}\alpha + \mathcal{O}(\alpha^3) \tag{4.14}$$

For $\psi = 0$, the limit indeed reduces to the matrix factorization obtained from the taking product of minimal models $M_1(x_1) \otimes M_1(x_2) \otimes M_1(x_3)$ and perturbing by the marginal operator $\Phi = \phi_0^1(x_i)^{\otimes i=1\dots3}$. Indeed, setting $\beta = 1$ for convenience, we find,

$$\tilde{g} \sim \begin{pmatrix} 0 & x_1 & x_2 & x_3 \\ x_1 & 0 & -x_3^2 & x_2^2 \\ x_2 & -x_3^2 & 0 & -x_1^2 \\ x_3 & -x_2^2 & x_1^2 & 0 \end{pmatrix} + \alpha \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & x_2x_3 & 0 & 0 \\ 0 & 0 & x_1x_3 & 0 \\ 0 & 0 & 0 & x_1x_2 \end{pmatrix} + \mathcal{O}(\alpha^2)\,. \tag{4.15}$$

Let us also write out the other matrix after the similarity transformation

$$\tilde{f} = \begin{pmatrix} \rho & \pi \\ \xi & A \end{pmatrix} \tag{4.16}$$

where

$$\rho = x_1x_2x_3\left(\frac{3(\gamma+\beta) - \alpha\psi}{\alpha^2}\right) \tag{4.17}$$

and

$$\pi = \xi^t = \left(x_1^2 + \frac{\beta+\gamma}{\alpha}x_2x_3, x_2^2 + \frac{\beta+\gamma}{\alpha}x_1x_3, x_3^2 + \frac{\beta+\gamma}{\alpha}x_1x_2\right). \tag{4.18}$$

Again, the limit $\alpha \to 0$ is smooth in view of (4.14), and is of the form

$$\tilde{f} \sim \begin{pmatrix} 0 & x_1^2 & x_2^2 & x_3^2 \\ x_1^2 & 0 & x_3 & -x_2 \\ x_2^2 & -x_3 & 0 & x_1 \\ x_3^2 & x_2 & -x_1 & 0 \end{pmatrix} + \alpha \begin{pmatrix} -x_1x_2x_3 & 0 & 0 & 0 \\ 0 & x_1 & 0 & 0 \\ 0 & 0 & x_2 & 0 \\ 0 & 0 & 0 & x_3 \end{pmatrix}. + \mathcal{O}(\alpha^2) \tag{4.19}$$

It is easy to see that the resulting $Q = \left(\begin{smallmatrix} 0 & \tilde{f} \\ \tilde{g} & 0 \end{smallmatrix}\right)$ is equivalent with the tensor product of minimal model branes $M_1(x_i)^{\otimes i=1,2,3}$ deformed by Φ. Indeed, using the formulae in subsection 3.2, we find

$$
\begin{aligned}
Q_1(x_1)+Q_1(x_2)+Q_1(x_3) &= \begin{pmatrix} 0 & x_1 \\ x_1^2 & 0 \end{pmatrix} \otimes 1 \otimes 1 + \sigma_1 \otimes \begin{pmatrix} 0 & x_2 \\ x_2^2 & 0 \end{pmatrix} \otimes 1 + \sigma_1 \otimes \sigma_2 \otimes \begin{pmatrix} 0 & x_3 \\ x_3^2 & 0 \end{pmatrix} \\
&= \begin{pmatrix}
0 & x_3 & x_2 & 0 & x_1 & 0 & 0 & 0 \\
x_3^2 & 0 & 0 & x_2 & 0 & x_1 & 0 & 0 \\
x_2^2 & 0 & 0 & -x_3 & 0 & 0 & x_1 & 0 \\
0 & x_2^2 & -x_3^2 & 0 & 0 & 0 & 0 & x_1 \\
x_1^2 & 0 & 0 & 0 & 0 & -x_3 & -x_2 & 0 \\
0 & x_1^2 & 0 & 0 & -x_3^2 & 0 & 0 & -x_2 \\
0 & 0 & x_1^2 & 0 & -x_2^2 & 0 & 0 & x_3 \\
0 & 0 & 0 & x_1^2 & 0 & -x_2^2 & -x_3^2 & 0
\end{pmatrix}. \qquad (4.20)
\end{aligned}
$$

Up to some reshuffling and CP signs, this is equivalent to (4.15), (4.19).

4.2 Moduli space of branes

We have seen that for any (α, β, γ) obeying

$$\alpha^3 + \beta^3 + \gamma^3 + \psi\alpha\beta\gamma = 0\,,$$

we have a 4×4 matrix factorization of W in (4.1). The scaling $(\alpha, \beta, \gamma) \to (u\alpha, u\beta, u\gamma)$ corresponds to a basis change of Chan-Paton factors. For example, in the region $\alpha\beta\gamma \neq 0$, the scaling corresponds to

$$f \to \begin{pmatrix} 1 & \\ & u\mathbf{1}_3 \end{pmatrix} f \begin{pmatrix} u^{-1} & \\ & \mathbf{1}_3 \end{pmatrix}, \qquad g \to \begin{pmatrix} u & \\ & \mathbf{1}_3 \end{pmatrix} g \begin{pmatrix} 1 & \\ & u^{-1}\mathbf{1}_3 \end{pmatrix},$$

which is done by a certain scaling of the basis elements of the Chan-Paton factor. It is a simple exercise to show this also in the regions near $\alpha\beta\gamma = 0$. Thus, matrix factorizations for a fixed ψ are parameterized by the torus of modulus ψ itself. Namely, the moduli space of the branes for a given torus is the torus itself!

Note that we need to make a basis change (4.12) as we approach the point $[\alpha, \beta, \gamma] = [0, 1, -1]$, and similarly near the eight other points with $\alpha\beta\gamma = 0$. This suggests that the supercharge Q is not a holomorphic function on the moduli space, but is a section of a certain bundle. In the context of string field theory, supercharge Q defines the BRST operator. This is an interesting circumstance where non-trivial gauge transformations of the BRST operator play an important role. We expect that this property holds for more general Gepner models including those with $\hat{c} = 3$ that are relevant for $\mathcal{N} = 1$ compactifications.

4.3 Geometric interpretation

As we have seen, the above family of branes are finite deformations of the brane $M_1(x_1) \otimes M_1(x_2) \otimes M_1(x_3)$ at $\psi = 0$, which is identified as the $\mathbf{L} = (0, 0, 0)$ rational brane in CFT.[1]

[1] We have been ignoring the label $m = 1, 3, 5$ that specifies the $\mathbb{Z}_3$ action on the Chan-Paton factor, but the above story holds in all values of m. They correspond to $\mathbf{L} = (0, 0, 0)$ and $M = m - 1$ branes.

The geometrical interpretation of such rational branes are studied in [27] by computing the charges. According to their results, it can be interpreted as a brane wrapped on the torus itself and supporting a holomorphic bundle of trivial topology. The brane is specified by the holomorphic structure of the line bundle. Thus, the moduli space is the jacobian of the torus, which is the same as the torus itself as a complex manifold. We have seen that this is indeed the case for the Landau-Ginzburg branes.

In [17], a direct way to obtain more precise geometry of the brane is described. The prescription of [17] is first to view a matrix factorization, as they were originally introduced in [35], as defining a free resolution of a so-called maximal Cohen-Macaulay module $\mathcal{M}$ over the ring $R = \mathbb{C}[x_i]/W$,

$$\cdots \xrightarrow{B} R^3 \xrightarrow{A} R^3 \xrightarrow{B} R^3 \xrightarrow{A} R^3 \longrightarrow \mathcal{M} \longrightarrow 0\,, \tag{4.21}$$

i.e., $\mathcal{M} = \mathrm{Coker}A$. The second step of [17] is to consider the sheafification of $\mathcal{M}$, after which it obtains a geometric interpretation as a bundle over the elliptic curve defined by the vanishing of the polynomial (4.1) in $\mathbb{P}^2$. This procedure is very reminiscent of the gauged linear sigma model philosophy. It appears to make sense in more general Gepner models as well, and it would be very important to verify it from the physics point of view.

5. The quintic

In this section, we consider the LG model based on the general quintic superpotential

$$W = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 + \text{deformations} \tag{5.1}$$

and orbifolded by a diagonal $\mathbb{Z}_5$. When all deformations vanish, we can obtain matrix factorizations by taking tensor products of minimal model branes. We are interested in studying the behavior of these factorizations under open and closed string deformations.

5.1 A simple example

Let us consider

$$M = M_1(x_1) \otimes M_1(x_2) \otimes M_1(x_3) \otimes M_1(x_4) \otimes M_1(x_5) \tag{5.2}$$

where the tensor product is taken in the graded sense. (M corresponds to the $\mathbf{L} = (0,0,0,0,0)$ Recknagel-Schomerus [36] brane in CFT.) After orbifolding, the open string spectrum of M consists of one bosonic operator (the identity), and one fermionic, which is given as a tensor product

$$\Phi = \phi^1_{1,1,j=0}(x_i)^{\otimes i=1,\ldots 5}\,. \tag{5.3}$$

In distinction to the case of the torus, Φ, which is the "Serre dual" of the identity, has $U(1)$ charge 3, and does not lead to a marginal deformation. So, M is rigid, and does not have any moduli space.

Instead, let us ask the question what happens to M under deformations of the bulk. Consider adding a degree 5 monomial to W,

$$W(\psi) = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 + \psi \prod x_i^{m_i} \tag{5.4}$$

with $\sum m_i = 5$. To be specific, we take $x_1x_2x_3x_4x_5$, but other cases work similarly. We are looking for a deformation

$$Q = Q_0 + \Delta Q \tag{5.5}$$

where Q_0 is obtained from taking the tensor product of M_1 in five minimal models.

$$Q_0 = Q_1(x_1) + Q_1(x_2) + Q_1(x_3) + Q_1(x_4) + Q_1(x_5)\,, \tag{5.6}$$

where

$$Q_1(x) = \begin{pmatrix} 0 & x \\ x^4 & 0 \end{pmatrix}, \tag{5.7}$$

and it is understood that $Q_1(x_i)$ only operates in the i-th factor in (5.2). Imposing $Q^2 = W(\psi)$ leads to the equation

$$\{Q_0, Q_1\} + (\Delta Q)^2 = \psi x_1x_2x_3x_4x_5 \tag{5.8}$$

A solution to this equation is

$$\Delta Q = \psi \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes x_2 \mathrm{id} \otimes \cdots \otimes x_5 \mathrm{id}\,. \tag{5.9}$$

Indeed

$$\left\{ Q_1(x_1), \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 0 & x_1 \\ x_1^4 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\} = x_1 \,\mathrm{id} \tag{5.10}$$

$$\big\{Q_1(x_2), \Delta Q\big\} = 0 \tag{5.11}$$

etc., and

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}^2 = 0\,. \tag{5.12}$$

Of course, we could have chosen a different solution

$$\widetilde{\Delta Q} = x_1 \mathrm{id} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \cdots \otimes x_5 \mathrm{id} \tag{5.13}$$

but ΔQ and $\widetilde{\Delta Q}$ are simply related by a gauge transformation.

$$\Delta Q - \widetilde{\Delta Q} = \left\{ Q_0, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes x_3 \mathrm{id} \otimes \cdots \otimes x_5 \mathrm{id} \right\}. \tag{5.14}$$

As mentioned above, an analogous deformation of the boundary exists for deformation of the bulk by any monomial of degree 5. Moreover, these deformations mutually (anti-)commute, so we obtain a factorization of a generic quintic superpotential which reduces for the Fermat quintic to the tensor product of minimal model branes.

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5.2 An obstructed deformation

Consider the brane

$$M = M_2(x_1) \otimes M_2(x_2) \otimes M_2(x_3) \otimes M_2(x_4) \otimes M_2(x_5) \tag{5.15}$$

which corresponds to the $\mathbf{L} = (1,1,1,1,1)$ boundary state in CFT. In a single minimal model with $k = 3$, the brane M_2 has two bosonic and two fermionic boundary fields with the following charges

$$\phi_0^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad q = 0 \tag{5.16}$$

$$\phi_1^1 = \begin{pmatrix} 0 & 1 \\ -x & 0 \end{pmatrix} \qquad q = \frac{1}{5} \tag{5.17}$$

$$\phi_1^0 = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \qquad q = \frac{2}{5} \tag{5.18}$$

$$\phi_0^1 = \begin{pmatrix} 0 & x \\ -x^2 & 0 \end{pmatrix} \qquad q = \frac{3}{5}\,. \tag{5.19}$$

Note that the spectrum of charges is identical to the bulk, where we have the (c, c) ring $(0, x, x^2, x^3)$ with the charges $q_L = q_R = (0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5})$, respectively.

Therefore, when we take tensor product and orbifold, the projections work in the same way in the bulk and on the boundary. In the ordinary $\mathbb{Z}_5$ orbifold, the brane M has 101 fermionic operators with charge $q = 1$, and 101 bosonic operators with charge $q = 2$ (and in addition one operator of charge 0 and one of charge 3, but they are not important here). In the $(\mathbb{Z}_5)^4$ orbifold, which is the mirror quintic, M has one fermionic operator

$$\Phi = \phi_1^1(x_i)^{\otimes i=1,\ldots,5} \tag{5.20}$$

and one bosonic operator, which is the Serre dual (2.19) of Φ,

$$\Phi^\dagger = \phi_1^0(x_i)^{\otimes i=1,\ldots,5} = x_1 x_2 x_3 x_4 x_5\, \mathrm{id}^{\otimes 5} \tag{5.21}$$

We note that $\Phi^\dagger$ is proportional to the identity matrix. It can be viewed either as a boundary field, where it corresponds to a charge $q = 2$ field in the cohomology of M, or as a bulk field, where it is an element of the (c, c) ring of left-right charge $(1, 1)$. Let us distinguish the bulk field by denoting it as Ψ. We note that

$$\Phi^2 = -\Phi^\dagger = -\Psi\, \mathrm{id}\,. \tag{5.22}$$

Consider deforming Q by Φ

$$Q \to Q(\varphi) = Q + \varphi\Phi\,. \tag{5.23}$$

$Q(\varphi)$ fails to square to W at order φ^2,

$$\big(Q(\varphi)\big)^2 - W = -\varphi^2 \Phi^\dagger \tag{5.24}$$

and we can ask whether we can improve $Q(\varphi)$ at order φ^2 in order to fix this. However, this would require a field $\tilde{\Phi}$ with the property

$$\{Q, \tilde{\Phi}\} = -\Phi^2 = \Phi^\dagger\,. \tag{5.25}$$

But such a field cannot exist, because $\Phi^\dagger$ is non-trivial in the cohomology of Q. We conclude that the deformation is obstructed at order φ^2.

On the other hand, if we consider deforming the bulk

$$W \to W(\psi) = W + \psi\Psi\,, \tag{5.26}$$

we can modify Q by Φ as in (5.23). Imposing $Q(\varphi)^2 = W(\psi)$ leads to the equation

$$\varphi^2 + \psi = 0. \tag{5.27}$$

This is the F-term equation of the system. For each $\psi \neq 0$, it has two solutions for φ, and we note immediately that the two solutions are not gauge equivalent because Φ is not exact.

5.3 Superpotential and a mirror interpretation

To give a geometric interpretation to (5.27), we need to recall the geometric objects that correspond to the LG branes we have been studying. The Gepner to large volume mapping of the charge lattice for B-type branes on the quintic was discussed in [27]. One of the 5 branes with $\mathbf{n} = (2,2,2,2,2)$ in the $\mathbb{Z}_5$ orbifold of (5.1) corresponds at large volume to a rank 8 bundle with Chern character $8 - 4H - 4H^2 + \frac{7}{3}H^3$, where H is the hyperplane of $\mathbb{P}^4$. The interest of this brane is that it is the anomaly cancelling bundle for type-I string theory on quintic [37] (with non-trivial action on the B-field). It would be interesting to check geometrically the fact that this bundle has 101 first order deformations, and that the one associated with $x_1x_2x_3x_4x_5$ is obstructed as in (5.27).

A different interpretation results by considering the factorization M in the mirror quintic, the $(\mathbb{Z}_5)^4$ orbifold of (5.1). There are 625 branes of this type, which via mirror symmetry correspond precisely to 625 rational A-type RS branes in the quintic model. As also shown in [27, 37], these 625 branes are identified at large volume with the 625 special lagrangian submanifolds obtained as real quintics. Our F-term constraint (5.27) provides a definite solution of a certain puzzle that has accompanied this identification. Topologically, a real quintic is nothing but an $\mathbb{RP}^3$, and as such is geometrically rigid. Nonetheless, there are two choices of flat $U(1)$ bundle coming from $\pi_1(\mathbb{RP}^3) = \mathbb{Z}_2$ [27]. These two supersymmetric branes for fixed Kähler class (which corresponds to ψ in this context) precisely match with the two solutions of (5.27). The fact that the brane on $\mathbb{RP}^3$ develops a massless field Φ as it is continued to small volume is something that is not predictible using classical geometry, and it would be interesting to understand how it comes about.

Let us make some comments on the spacetime superpotential $\mathcal{W}$. It must be a holomorphic function of (ψ, φ) such that the solution to $\mathcal{W} = \partial_\psi\mathcal{W} = \partial_\varphi\mathcal{W} = 0$ is given by (5.27). It must also be invariant under the $\mathbb{Z}_5$ identification, $(\psi,\varphi)\to(\alpha\psi,\alpha^{-2}\varphi)$, $\alpha^5=1$, that is induced from the change of variables $(x_1,x_2,x_3,x_4,x_5)\to(\alpha x_1,x_2,x_3,x_4,x_5)$. One satisfying these constraints is $\mathcal{W}=f((\psi+\varphi^2)^5)$, where $f(x)$ is a function such that $f(x)=f'(x)=0$ has no solution except possibly $x=0$. Note that this form of the superpotential is consistent with the fact that both ψ and φ are massless at $\psi=\varphi=0$. There is another, global constraint — the superpotential is a section of a line bundle $\mathcal{L}$ over moduli space determined from the Kähler class [38].[2] Thus, it is important to know the global structure of the moduli space as well as the complex structure of the line bundle $\mathcal{L}$. It would be interesting if the non-trivial global structure of the supercharge Q on the moduli space, as the one we have seen in the torus example (section 4.2), has something to do with the line bundle $\mathcal{L}$. We hope to clarify this point in a future work.

6. Other models

In this final section, we illustrate in two further examples some other general features discussed in section 2. We will see that the obstruction can generically appear at higher order in the perturbation, that there can be unobstructed boundary deformations also in the case $\hat{c}=3$, as well as the fact that the presence of D-branes can obstruct marginal bulk deformations.

6.1 Obstruction of bulk deformations by D-branes

Consider the "two-parameter" model $\mathbb{P}_{1,1,2,2,2}[8]$. The LG potential is

$$W=x_1^8+x_2^8+x_3^4+x_4^4+x_5^4+\text{deformations}\,. \tag{6.1}$$

For simplicity, we will consider the orbifold of (6.1) by the maximal Greene-Plesser orbifold group, $\Gamma=\mathbb{Z}_8\times\mathbb{Z}_8\times\mathbb{Z}_4\times\mathbb{Z}_4$. This leaves only two marginal bulk deformations

$$\begin{aligned}\Psi_1&=x_1x_2x_3x_4x_5\\ \Psi_2&=x_1^4x_2^4\,.\end{aligned} \tag{6.2}$$

The branes associated with the factorization (for simplicity, we again omit the labels for the orbifold group action),

$$M=M_2(x_1)\otimes M_2(x_2)\otimes M_2(x_3)\otimes M_2(x_4)\otimes M_2(x_5) \tag{6.3}$$

shows some interesting properties. There is only one marginal operator invariant under Γ,

$$\begin{aligned}\Phi&=\phi_1^1(x_i)^{\otimes i=1,\ldots,5}\\ &=\begin{pmatrix}0&1\\-x_1^4&0\end{pmatrix}\otimes\begin{pmatrix}0&1\\-x_2^4&0\end{pmatrix}\otimes\begin{pmatrix}0&1\\-1&0\end{pmatrix}\otimes\begin{pmatrix}0&1\\-1&0\end{pmatrix}\otimes\begin{pmatrix}0&1\\-1&0\end{pmatrix}\end{aligned} \tag{6.4}$$

[2] As is common practice, we are here using the term "moduli space" even though these fields are moduli only before imposing the constraint from the superpotential.

with conjugate obstruction

$$\Phi^\dagger = \Psi_1 \cdot \mathrm{id} \tag{6.5}$$

and

$$\Phi^2 = -\Psi_2 \cdot \mathrm{id}\,. \tag{6.6}$$

One easily checks that Φ^2 (and with it the marginal bulk deformation Ψ_2) are exact on the boundary. The deformation problem therefore has a solution up to second order. When constructing the solution to higher order, as in (2.23), one soon notices that Q_n is only non-trivial in the first two minimal model factors, whereas in the last three factors (those with $h_i = 4$), it is either proportional to the identity or to ϕ_1^1, which are both independent of x_3, x_4, x_5. Thus, the obstruction $\Phi^\dagger = \Psi_1$ can actually never appear in the perturbative series, and we conclude that the first order deformation by Φ is unobstructed. One can also construct the finite deformation explicitly (for arbitrary ψ_2), much as in the case of the torus.

The natural question that then arises is "What does the obstruction $\Phi^\dagger$ actually obstruct?" (We here mean "natural" in the sense of naturalness in QFT. This does not imply that the question must have an answer.) We claim that it actually expresses the fact that, in the presence of M, Ψ_1 is not an allowed bulk deformation anymore. To justify the claim, we have to show that the equation

$$\{Q, A\} + A^2 = \psi_1 \Psi_1 \,\mathrm{id} = \psi_1 x_1 x_2 x_3 x_4 x_5 \,\mathrm{id} \tag{6.7}$$

where Q is the supercharge corresponding to M and A is an arbitrary fermionic field with charge 1, has no solution when $\psi_1 \neq 0$.

Strictly speaking, we should emphasize that showing that (6.7) has no solution does not exclude the possibility that we can find a solution of $P^2 = W$ for non-zero ψ_1 which for $\psi_1 \to 0$ does not reduce to the tensor product solution Q, but is still continuously connected to it via massive deformations. In other words, the moduli space of our brane could have several branches at $\psi_1 = 0$, around some of which Ψ_1 is a valid deformation. We will here only consider the problem around the rational point.

To show that $x_1 x_2 x_3 x_4 x_5 \cdot \mathrm{id}$ is not contained in the l.h.s. of (6.7), we expand A as

$$A = \sum_j \alpha_j A^j \,, \tag{6.8}$$

where the A^j form a basis of Γ-equivariant fermionic maps in $\mathrm{Hom}(M, M)$ with homogeneous degree $q = 1$. Note that even without going to the kernel of Q, this space is finite dimensional.

Since Q is at least quadratic in the x_i's, the first term in (6.7) cannot contain Ψ_1. To check the second term, we notice that we can focus on those A^j in (6.8) which are at most linear in the x_i's. Imposing in addition invariance under Γ and a total R-charge of 1 in fact leaves only very few possibilities. Basis elements A^j must be of one of the forms

$$A^1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a_3 \\ b_3 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a_4 \\ b_4 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a_5 \\ b_5 & 0 \end{pmatrix} \tag{6.9}$$

or

$$A^2 = \begin{pmatrix} 0 & 0 \\ x_1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ x_2 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & c_3x_3 \\ d_3x_3 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & c_4x_4 \\ d_4x_4 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & c_5x_5 \\ d_5x_5 & 0 \end{pmatrix} \tag{6.10}$$

where a_i, b_i, c_i and d_i are arbitrary scalars. In the second term on the l.h.s. of (6.7), this results in expressions of the form

$$\{A^1, A^2\} = x_1x_2x_3x_4x_5 \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right] \otimes \cdots, \tag{6.11}$$

which clearly falls short of $\Psi_1 \cdot \mathrm{id}$.

To summarize, the sole F-term constraint in the present model is

$$\psi_1 = 0. \tag{6.12}$$

6.2 Obstruction at higher order

Finally, we consider a slightly more complicated model which combines several of the previous features.

The model under consideration is a popular "three-parameter" model with

$$W = x_1^{15} + x_2^5 + x_3^5 + x_4^5 + x_5^3 + \text{deformations} \tag{6.13}$$

and where we again orbifold by the maximal group $\mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_3$. Marginal bulk deformations are

$$\begin{aligned} \Psi_1 &= x_1x_2x_3x_4x_5 \\ \Psi_2 &= x_1^{10}x_5 \\ \Psi_3 &= x_1^6x_2x_3x_4\,. \end{aligned} \tag{6.14}$$

The tensor product brane with $\mathbf{n} = (7, 2, 2, 2, 1)$ has three invariant marginal boundary deformations

$$\begin{aligned} \Phi_1 &= \begin{pmatrix} 0 & 1 \\ -x_1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -x_2 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -x_3 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -x_4 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -x_5 & 0 \end{pmatrix} \\ \Phi_2 &= x_1^5\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes \begin{pmatrix} 0 & 1 \\ -x_5 & 0 \end{pmatrix} \\ \Phi_3 &= x_1^3\mathrm{id} \otimes \begin{pmatrix} 0 & 1 \\ -x_2 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -x_3 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -x_4 & 0 \end{pmatrix} \otimes \mathrm{id}\,. \end{aligned} \tag{6.15}$$

The obstructions are

$$\begin{aligned} \Phi_1^\dagger &= x_1^6x_2x_3x_4\,\mathrm{id} \\ \Phi_2^\dagger &= \begin{pmatrix} 0 & x_1 \\ -x_1^2 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & x_2 \\ -x_2^2 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & x_3 \\ -x_3^2 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & x_4 \\ -x_4^2 & 0 \end{pmatrix} \otimes \mathrm{id} \\ \Phi_3^\dagger &= \begin{pmatrix} 0 & x_1^3 \\ -x_1^4 & 0 \end{pmatrix} \otimes x_2\mathrm{id} \otimes x_3\mathrm{id} \otimes x_4\mathrm{id} \otimes \begin{pmatrix} 0 & 1 \\ -x_5 & 0 \end{pmatrix}. \end{aligned} \tag{6.16}$$

Also,

$$\begin{aligned}
\Phi_1^2 &= -\Psi_1 \mathrm{id} \\
\Phi_2^2 &= -\Psi_2 \mathrm{id} \\
\Phi_3^2 &= -\Psi_3 \mathrm{id} = -\Phi_1^\dagger \,.
\end{aligned} \tag{6.17}$$

Thus Φ_3 behaves much like the obstructed deformation we studied in the quintic case. Φ_2^2 or Ψ_2 are exact on the boundary, and the obstructions all involve x_2, x_3, x_4 non-trivially, so this case is similar to the one in the previous subsection, and Φ_2 is not obstructed. Consider, finally, deformations by $Q_1 = \Phi_1$. To second order, we can write,

$$\Phi_1^2 = \left\{ Q, \underbrace{-x_1 \mathrm{id} \otimes x_2 \mathrm{id} \otimes x_3 \mathrm{id} \otimes x_4 \mathrm{id} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{=:Q_2} \right\} \tag{6.18}$$

We then find

$$\{Q_2, Q_1\} = \Phi_2^\dagger \tag{6.19}$$

so that here the obstruction appears at third order in the deformation. One may also note that the three first order deformation Φ_1, Φ_2 and Φ_3 do not mutually commute. As a consequence, their joint deformation problem is more intricate.

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